Discontinuous Functions as Limits of Compactly Supported Formulas

J. Marshall Ash

J. Marshall Ash (mash@depaul.edu, MR ID 27660, ORCID 0000-0003-3053-0988) is professor emeritus of mathematics at DePaul University. He received his Ph.D. in mathematics from the University of Chicago in 1966. He taught mathematics at DePaul University from 1969 until his retirement 45 years later. He continues to do collaborative mathematical research. He enjoys weekly 25 mile bicycle rides with friends in the Berkshire hills of western Massachusetts; reading some of the classics that were always put off for "later"; and viewing movies, plays and occasional Yankee games with his family.

In beginning textbooks, bounded real valued functions with domain **R** and one point of discontinuity are usually defined piecewise with each piece being given by a formula. When I first took calculus, the only functions I was familiar with were those associated with a formula. It bothered me that when a counterexample with a discontinuity was called for, the machinery of cases was used, being introduced for the first time at exactly this point. (See, for example, any of references [**1**–**4**].) Even though such examples are satisfactory from the modern point of view, this paper shows that it is not very hard to create many examples of functions with a single point of discontinuity while completely avoiding functions given by means of cases.

Bounded functions with exactly one point of discontinuity can be discontinuous in six ways. Here we give six examples, each having a different type of discontinuity at its unique point of discontinuity. Each example type will be represented as a pointwise limit of quite simple continuous functions. The approximating functions will be given by elementary formulas not requiring piecewise presentation. Furthermore, these functions will all have compact support. A function has *compact support* if the set of points where it is non-zero is contained in some finite interval.

We avoid cases, but the price paid for this is that we require the concept of a pointwise convergent sequence. Thus this article is appropriate for an undergraduate introductory analysis course, but may be too hard for most beginning calculus courses. Functions of compact support are usually encountered only in more advanced analysis courses where they are often built from C^∞ pieces and used for localizing functions, for example for creating a partition of unity. So it may come as a surprise to some that there exist *any* compactly supported function expressible as a simple formula.

Here are three examples of real valued functions of a real variable that are discontinuous at $x = 0$. All three are defined piecewise, that is to say, by cases.

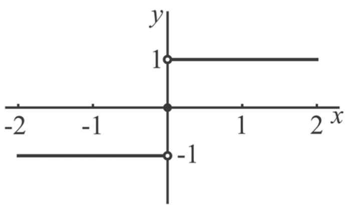
$$\operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

$$\chi(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$$

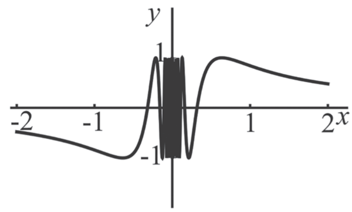
doi.org/10.1080/07468342.2020.1820284
MSC: 26A15, 26A21; 26A03, 26A06, 26A09

$$s(x) = \begin{cases} \sin \dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

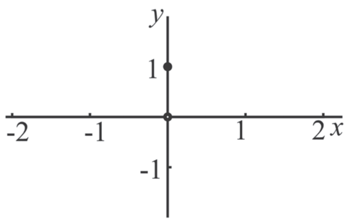
These three functions, together with simple combinations of them, give a fairly complete picture of the six ways a bounded function can be discontinuous at a point. The grid in Figure 1 shows the six ways a single point discontinuity can occur. Each example has a formula depending only on $\operatorname{sgn}(x)$, $\chi(x)$, and $s(x)$.



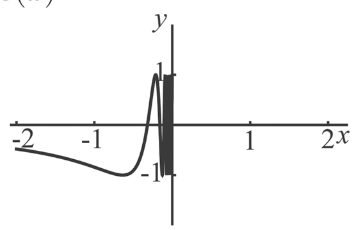
(a) Jump with left limit, right limits, values all distict: $sgn(x)$



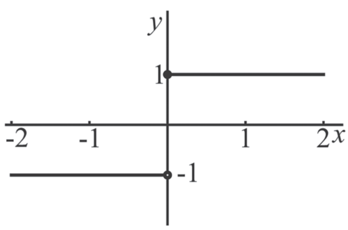
(b) Limits from both sides do not exist: $s(x)$



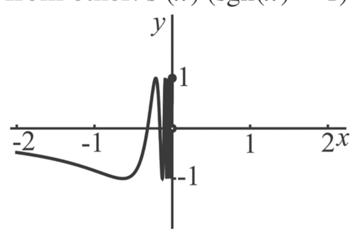
(c) Limit and value both exist, but are not equal: $\chi(x)$



(d) No limit from one side, continuous from other: $s(x)(\operatorname{sgn}(x) - 1)/2$



(e) Jump, but continuous from one side: $\operatorname{sgn}(x) + \chi(x)$



(f) No limit from one side, value $\neq$ other side limit: $s(x)(\operatorname{sgn}(x) - 1)/2 + \chi(x)$

Figure 1. The 6 types of discontinuities.

We see from these six examples that every possible type of discontinuity can be expressed in terms of three functions, $\operatorname{sgn}(x)$, $\chi(x)$, and $s(x)$. This paper demonstrates that $\operatorname{sgn}(x)$, $\chi(x)$, and a third function $S(x)$ may be written as limits of sequences of continuous functions. The function $S(x)$ will be defined later in this paper and will be used in place of $s(x)$ since it will share with $s(x)$ the property of having neither one sided limit at $x = 0$, but the elements of its approximating sequence will be simpler than trigonometric. All of the functions in the three approximating sequences have elementary formulas defined for all real numbers.

 © THE MATHEMATICAL ASSOCIATION OF AMERICA

sgn (x) and $\chi(x)$ as limits of sequences

Let

$$u(x) = \frac{x}{|x|+1}. \tag{1}$$

Then

$$\operatorname{sgn}(x) = \lim_{n\to\infty} u(nx). \tag{2}$$

Write $|x|$ as $\sqrt{x^2}$, to see that there are no hidden piecewise defined objects here.
Let

$$v(x) = \frac{1}{|x|+1}. \tag{3}$$

Then

$$\chi(x) = \lim_{n\to\infty} v(nx). \tag{4}$$

Graph $u(100x)$ and $v(100x)$ to get some intuition about these two examples. Formally, whenever we write

$$f(x) = \lim_{n\to\infty} f_n(x),$$

we mean that f is the *pointwise limit* of the sequence of functions $\{f_n\}_{n\geq 1}$, i.e., that whenever any real number x is fixed, the resulting sequence of numbers $\{f_n(x)\}$ tends to the limiting number $f(x)$ as n tends to infinity.

For each positive integer n, both the functions $u(nx)$ and $v(nx)$ are continuous and expressed as elementary formulas with domain **R**.

Unfortunately, they do not have compact support. The next section creates a tool that will allow us to deal with this issue.

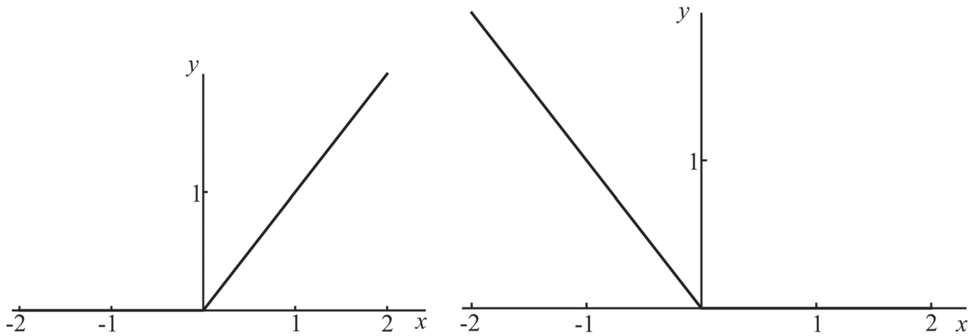
Bumps

We will build $S(x)$, our simpler version of $s(x)$, by adding together a set of bumps. A bump will be a very simple function having compact support. Recall that a function has compact support if the set of points where it is non-zero is contained in some finite interval.

Let $p(x) = x^+ = (|x|+x)/2$ and $n(x) = x^- = (|x|-x)/2$. The graphs of p and n are shown in Figure 2.

The product $p(x-a)\,n(x-b)$ is positive on the interval (a, b) and is zero on the complement of (a, b). On (a, b) it agrees with the quadratic $(x-a)(b-x)$ which achieves a maximum value of $\left(\frac{b-a}{2}\right)^2$ at the midpoint $x = \frac{a+b}{2}$. We normalize to create a non-negative function of maximum height 1. Our *bumps* are the family of functions, one for each pair of real numbers a, b with $a < b$, given by

$$B_{a,b}(x) = \left(\frac{2}{b-a}\right)^2 p(x-a)\,n(x-b)$$



(a) The graph of $p(x)$ (b) The graph of $n(x)$

Figure 2. The graphs of p and n.

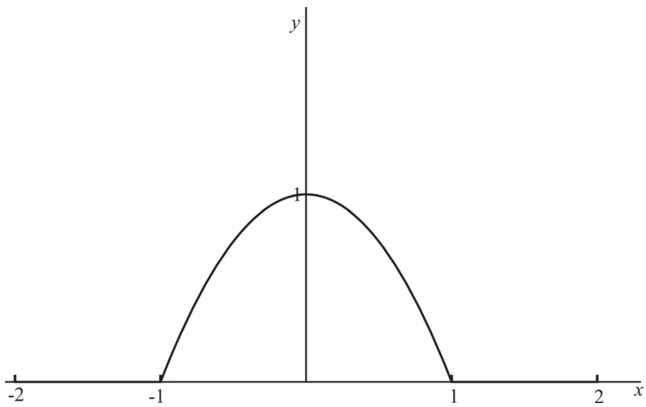


Figure 3. The simple bump graph $B_{-1,1}(x)$.

So $B_{a,b}$ can be expressed as the formula

$$B_{a,b}(x) = \left(\frac{1}{b-a}\right)^2 (|x-a| + (x-a))(|x-b| - (x-b)). \tag{5}$$

For another, more geometrically based formula: Recall that for x in the interval $[a, b]$, $B_{a,b}(x)$ is the quadratic passing through the endpoints and having maximum value 1 at the center $m = (b+a)/2$, so if we set $\delta = (b-a)/2$ to the half-length of the interval, we get the alternative formula

$$\begin{aligned} B_{a,b}(x) &= \left(1 - \left(\frac{x-m}{\delta}\right)^2\right)^+ \qquad (6) \\ &= \frac{1}{2}\left(\left|1 - \left(\frac{x-m}{\delta}\right)^2\right| + \left(1 - \left(\frac{x-m}{\delta}\right)^2\right)\right). \end{aligned}$$

The graph of the bump $B_{-1,1}(x)$ is shown in Figure 3.

 © THE MATHEMATICAL ASSOCIATION OF AMERICA

For $n \geq 2$ and $|x| \leq \sqrt{n}$, $1 \geq B_{-n,n}(x) \geq B_{-n,n}\left(\sqrt{n}\right) = 1 - \frac{1}{n}$. It follows that for every real number x,

$$\lim_{n\to\infty} B_{-n,n}(x) = 1. \tag{7}$$

Examples as limits of compactly supported functions

Before we use bumps to create the example $S(x)$ mentioned above, we will use them to easily express both $\operatorname{sgn}(x)$ and $\chi(x)$ as limits of sequences $\{U_n(x)\}$ and $\{V_n(x)\}$ of functions which are not only continuous and expressed as elementary formulas with domain $\mathbf{R}$, but are also compactly supported.

The bump functions allow us to convert any example of a function discontinuous at a point being a limit of everywhere defined formulas into a similar example where the approximating functions are also compactly supported. For example, let $U_n(x)$ be $u(nx)\, B_{-n,n}(x)$. Formulas (1) and (6) show that for each integer n, $U_n(x)$ can be given by an everywhere defined, continuous, compactly supported, elementary formula; then limits (2) and (7) lead to

$$\begin{aligned}\lim_{n\to\infty} U_n(x) &= \lim_{n\to\infty} u(nx)\, B_{-n,n}(x) = \lim_{n\to\infty} u(nx) \lim_{n\to\infty} B_{-n,n}(x)\\ &= \operatorname{sgn}(x)\cdot 1 = \operatorname{sgn}(x).\end{aligned}$$

A very similar argument using (3) and (4) in place of (1) and (2) gets the same result for the discontinuous function $\chi(x)$. Another, even faster, way of treating χ is to write

$$\chi(x) = \lim_{n\to\infty} \left(B_{-1,1}(x)\right)^n.$$

In view of the formulas expressing all six types of discontinuous functions in terms of sgn, χ, and s, it suffices to produce a function $S(x)$ that, like $s(x)$, fails to have either one sided limit at $x = 0$; but that, unlike $s(x)$, is simply and naturally given as a limit of continuous functions, each of which is compactly supported and defined everywhere by an elementary formula.

To this end, fix a positive integer n and define $C_n(x) = B_{\frac{3}{4}\frac{1}{2^n},\frac{5}{4}\frac{1}{2^n}}\left(x - 1/2^n\right)$. Then the following three properties hold:

1. $C_n(x)$ is zero outside of $I_n = \left(\frac{3}{4}\frac{1}{2^n}, \frac{5}{4}\frac{1}{2^n}\right)$,
2. C_n is a quadratic bump of maximum height 1 at the center point $x = \frac{1}{2^n}$ of I_n and of height 0 at both endpoints of I_n, and
3. C_n is given by the everywhere defined formula (5) with $a = \frac{3}{4}\frac{1}{2^n}$ and $b = \frac{5}{4}\frac{1}{2^n}$.

The sum of the first N bumps,

$$g_N(x) = \sum_{n=1}^{N} C_n(x),$$

is an everywhere defined formula and has the value 0 at every point of the set $\left(-\infty, \frac{3}{4}\frac{1}{2^N}\right) \cup \left(\frac{5}{4}\frac{1}{2^1}, \infty\right)$. The bumps do not overlap, since C_{n+1} lies strictly to the left of C_n; in fact, the supremum of the set where $C_{n+1} \neq 0$ is $\frac{5}{4}\frac{1}{2^{n+1}}$ and the infimum of the set where $C_n \neq 0$ is $\frac{3}{4}\frac{1}{2^n}$ and

$$\frac{5}{4}\frac{1}{2^{n+1}} = \frac{5}{8}\frac{1}{2^n} < \frac{6}{8}\frac{1}{2^n} = \frac{3}{4}\frac{1}{2^n}.$$

Now let

$$g(x) = \lim_{N\to\infty} g_N(x) = \sum_{n=1}^{\infty} C_n(x).$$

It is easy to see that $g\left(\frac{1}{2^n}\right) = 1$ and that $g\left(\frac{3}{4}\frac{1}{2^n}\right) = 0$ for $n = 1, 2, 3, \ldots$. Thus $\lim_{x\to 0^+} g(x)$ does not exist. The graph of $g_6(x)$ is shown in Figure 4.

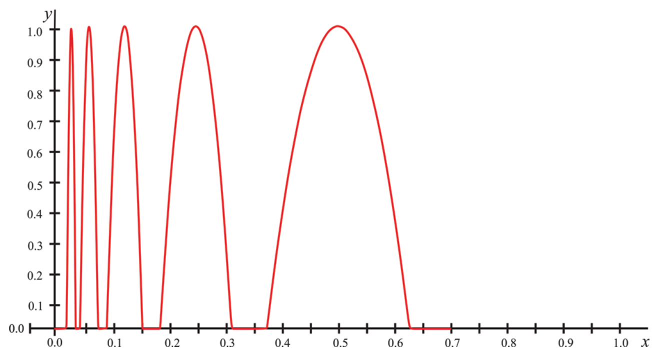


Figure 4. The graph of $g_5(x)$: a sum of bumps.

The function $g(x)$ does satisfy $\lim_{x\to 0^-} g(x) = g(0) = 0$, since $g(x) = 0$ for all non-positive x. For a function with neither left nor right limit at $x = 0$, use $S(x) = g(x) - g(-x)$. Note that $g(x)$ itself is also directly an instance of type 4 in our counterexample grid.

Remark. The continuous functions appearing in my three approximating sequences are all built by starting from x, x^2, $1/x$, and $\sqrt{x^2}$ and performing a few simple algebraic operations.

Related examples and exercises

A way to create the function $s(x)$ itself without cases is this. First let

$$s_n(x) = \sin\left(\frac{1}{|x| + \frac{1}{n\pi}}\right).$$

 © THE MATHEMATICAL ASSOCIATION OF AMERICA

As $n \to \infty$, at every x this sequence of functions approaches

$$s(|x|) = \begin{cases} \sin\left(\frac{1}{|x|}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}. \tag{8}$$

Since this function is even, we must multiply by $\operatorname{sgn}(x)$ to achieve $s(x)$ itself. The desired approximating sequence is $\{s_n(x)\, u(nx)\, B_{-n,n}(x)\}$; using (8), (2), and (7), we get

$$\lim_{n\to\infty} s_n(x)\, u(nx)\, B_{-n,n}(x) = s(|x|) \cdot \operatorname{sgn}(x) \cdot 1 = s(x).$$

The only weakness to this approach is that the approximating functions are trigonometric, and thus not very elementary.

Note that the bump function $B_{a,b}(x)$ has corners. If we replace $p(x)$ by

$$\frac{1}{2}(|x| + x)\, |x|^{n-1}$$

and then repeat the rest of the construction in a very similar way, the resulting analog of $B_{a,b}(x)$ will have a continuous derivative of order $n - 1$. I don't see an equally simple way to make the analogue of $B_{a,b}(x)$ be C^∞. From the point of view of this paper, the trouble with using $P(x) = \frac{1}{2}(|x| + x)\frac{1}{e^{\frac{1}{x^2}}}$ and $N(x) = \frac{1}{2}(|x| - x)\frac{1}{e^{\frac{1}{x^2}}}$ in place of $p(x)$ and $n(x)$ is that they are undefined when $x = 0$. However, if we are going to use the test of being acceptable to 18th century mathematicians, then removable discontinuities may be removed: thus since $\lim_{x\to 0} e^{-1/x^2} = 0$, they would take $e^{-1/0^2}$ to be zero. Under these rules, there does exist a fairly simple formula for a compactly supported C^∞ bump, namely $P(x - a)\, N(x - b)$ where $a < b$.

Here is an easy exercise for the reader: Start from this example and copy the procedure of this paper to find examples for all six types of discontinuities in our grid, using pointwise limits of sequences of compactly supported formulas which are better than ours in being C^∞, but worse than ours in requiring the non-elementary formula element e^{-1/x^2}.

If we relax the restriction that our functions be bounded and extend our study to include *all* real valued function with domain $\mathbf{R}$ and one point of discontinuity, then instead of the six types of behavior shown in Figure 1, there are 24 types of possible behavior at the discontinuity. Types (a) through (f) expand respectively to 3, 7, 2, 3, 2, and 7 distinct types. Two examples: type (a), a jump with left limit, right limit, and value all distinct, expands to these 3 types: both limits finite, one limit finite but the other limit $= +\infty$, and one limit $= +\infty$ and the other limit $= -\infty$; and type (f), no limit from one side, value distinct from the other sides limit, expands to these 7 types: (1) one side oscillates finitely and the finite limit on the other side is distinct from the value, (2) one side oscillates finitely and the limit of the other side is $+\infty$, (3) on one side the limsup is $+\infty$ and the liminf is finite and the finite limit on the other side is distinct from the value, (4) on one side the limsup is $+\infty$ and the liminf is finite and the limit on the other side is $+\infty$, (5) on one side the limsup is $+\infty$ and the liminf is finite and the limit on the other side is $-\infty$, (6) on one side the limsup is $+\infty$ and the liminf is $-\infty$ and the finite limit on the other side is distinct from the value, and (7) on one side the limsup is $+\infty$ and the liminf is $-\infty$ and the limit on the other side is $+\infty$.

More exercises: (1) Complete the classification of functions with one point of discontinuity. In other words, enumerate the 14 cases not spelled out above. (2) Then provide a simple example for each of the 18 types not covered in this paper, first given by cases, and then given by means of a sequence of continuous, simple, compactly supported formulas. The second part is pretty tedious, you might only want to do a few.

Acknowledgments. Fernando Gouvêa of Colby College showed me the bump function $B_{a,b}(x)$. My brother, Peter Ash of Cambridge College, told me that he had seen the function $\left(1-x^2\right)^+$ when he worked in computer graphics. This motivated the alternate formula (6) for the bump $B_{a,b}$. My colleague, Alan Berele, created $\{s_n(x)\}$ for me. I once wanted to put a graph of $\operatorname{sgn}(x)$ into a research paper using a graphing program that I had not sufficiently mastered to input piecewise defined functions. A colleague, Stephen Vági, suggested that I use $\frac{2}{\pi}\arctan(100x)$. His suggestion motivated my first sequence $\{u(nx)\}$ above. This paper was improved by two College Math. J. referees. My son, Michael Ash, improved the graphics.

Summary. A bounded real valued function with domain **R** and one point of discontinuity can be discontinuous in six ways. In beginning textbooks such functions are usually defined piecewise with each piece being given by a formula. Here we give six examples, each having a different type of discontinuity at its unique point of discontinuity. Each example type is represented as a pointwise limit of quite simple continuous functions. Each approximating function can be given by an elementary formula and also can be chosen to be of compact support.

References

[1] Courant, R., John, F. (1965). *Introduction to Calculus and Analysis*, Vol. 1. New York: Wiley, p. 31.
[2] Protter, M. H., Morrey, Jr., C. B. (1964). *Calculus With Analytic Geometry*, 1st ed. Reading, MA: Addison-Wesley, pp. 108–111.
[3] Salas, S. L., Hille, E. (1974). *Calculus, One and Several Variables*, Part 1, 2nd ed. Lexington, MA: Xerox, Section 2.
[4] Thomas, G. B. (1972). *Calculus and Analytic Geometry*, alternate ed. Reading, MA: Addison-Wesley, pp. 99–105.

 © THE MATHEMATICAL ASSOCIATION OF AMERICA